Eastward Deflection of a Particle Dropped from a Height: Description from the Inertial Frame of Reference

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Abstract

The phenomenon of eastward deflection of a particle dropped on the earth from a height is usually explained in terms of the effect of the Coriolis force on the moving particle as seen by an observer stationed on the surface of the earth. In this article we have tried to explain the same phenomenon from the point of view of an inertial observer.

A particle is dropped from the top of a tower. The point of release is at a height 'H' from the surface of the earth (Fig. 1). With respect to an inertial observer the earth is rotating about its axis with an angular velocity 'ω' and the particle falls under the attractive gravitational field of the earth. For the earth-bound observer this is a problem of motion under pseudo forces, in particular the Coriolis force and one can calculate the sideways deflection of the particle when it reaches the ground[1]. However, for an inertial observer the usual technique of handling the motion under central force is to consider it to be along an ellipse with earth's centre as a focus. Rees (1986)[2] has discussed this problem.

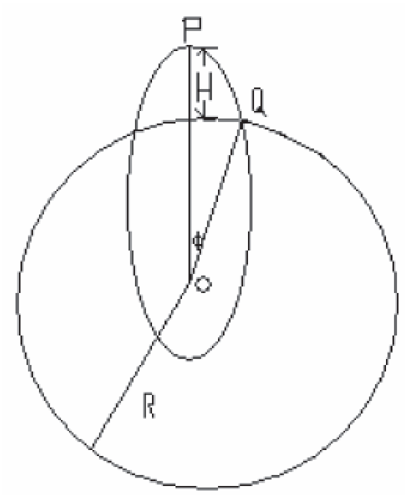

Fig. 1. Earth is assumed to be a homogeneous sphere and the drawing is not to scale for the sake of proper illustration of the trajectory of the particle.

The inertial observer describes the trajectory in terms of the following equation

$$r = \frac{l}{1 + \varepsilon \cos\theta}$$

where 'l' is the length of the semi latus-rectum and 'ε' the eccentricity of the elliptical path. Let us suppose, as shown in the figure, that the tower is in the equatorial plane of the earth while the latter is a homogeneous sphere of radius 'R' and mass 'M'.

The particle of mass 'm' after its release from the point P (as shown) reaches, after a time 'T', the point Q on the surface of the earth. During the time of flight the bottom of the tower will move to some new position which does not coincide with the point of intersection of the ellipse with the surface of the earth; when this happens, we say that there is sideways deflection. The time of flight

$$T = \frac{area\, OPQ}{areal\ velocity}$$

where, O is the centre of the earth. Under central force the areal velocity is constant i.e.

$$r^2\dot{\theta} = (R+H)^2\,\omega$$

(r, θ are the coordinates of the particle at any instant on the ellipse).

Therefore, $$\frac{d\theta}{dt} = \frac{(R+H)^2\,\omega}{r^2}$$

$$= \frac{(R+H)^2\,\omega(1+\varepsilon\cos\theta)^2}{l^2}$$

And $$T = \frac{l^2}{(R+H)^2\,\omega}\int_{\pi}^{\pi+\phi}\frac{d\theta}{(1+\varepsilon\cos\theta)^2} \qquad \text{(refer figure)}$$

(Note that the initial position corresponds to $\theta=\pi$, that is, the apogee of the path.) If we change the angular variable from θ to $\psi=\theta-\pi$, we get

$$T = \frac{l^2}{(R+H)^2\,\omega}\int_0^{\phi}(1-\varepsilon\cos\psi)^{-2}\,d$$

$$= \frac{l^2}{(R+H)^2\,\omega}\int_0^{\phi}\left[1-\varepsilon\left(1-\frac{\psi^2}{2}\right)\right]^{-2} \qquad (\psi \text{ is small})$$

Finally, $$T = \frac{l^2(1-\varepsilon)^{-2}}{(R+H)^2\,\omega}\left[\phi-\frac{1}{3}\frac{\varepsilon}{1-\varepsilon}\phi^3\right]$$

Initially i.e. at the point of release, $r = R + H$ & $\theta=\pi$.

Therefore, $$R+H = \frac{l}{1-\varepsilon}$$

Therefore, $$T = \frac{1}{\omega}\left[\phi-\frac{1}{3}\frac{\varepsilon}{1-\varepsilon}\phi^3\right] \tag{1}$$

Now, from usual kinematics under central force, we know

$$\varepsilon = \sqrt{1+\frac{2Eh^2}{m(GM)^2}}$$

where, E is the energy, h is the magnitude of the angular momentum per unit mass and G is the universal gravitational constant.

Again,

$$E = \frac{1}{2}m(R+H)^2\,\omega^2 - \frac{GMm}{(R+H)};$$

$$h = (R+H)^2\,\omega \quad \& \quad GM = gR^2$$

Therefore,

$$\varepsilon = \sqrt{1+\frac{2\left[\frac{(R+H)^2\omega^2}{2}-\frac{gR^2}{R+H}\right]}{g^2R^4}}$$

Expanding the polynomials and retaining terms which are linear and quadratic in ω and only linear in H (since ω is small and H<<R), we get

$$\varepsilon = \sqrt{1-\frac{2\left(R\omega^2+3H\omega^2\right)}{g}} \simeq 1-\dots \tag{2}$$

i.e. $$1-\varepsilon = \frac{(R+3H)\,\omega^2}{g} \tag{3}$$

Now, $R+H = \frac{l}{1-\varepsilon}$ and

$$R = \frac{l}{1+\varepsilon\cos(\pi+\phi)} \simeq \frac{l}{1-\varepsilon\left(1-\dots\right)}$$

Therefore, $$\frac{R}{R+H} = \frac{1-\varepsilon}{1-\varepsilon+\frac{\varepsilon\phi^2}{2}}$$

Thus, after simplification

$$\phi = \sqrt{\frac{2H(1-\varepsilon)}{R\varepsilon}} \tag{4}$$

Using (2) & (3) in (4), we get

$$\phi = \sqrt{\left\{ 2H \frac{(R+3H)\,\omega^2}{g} \right\} \Big/ \left\{ R \left(1 \right.\right.}$$

$$\simeq \sqrt{\frac{2H\omega^2}{g}\left(1 + \frac{R\omega^2}{2g} + 3\left(\frac{1}{R} + \frac{\omega^2}{2g} \right.\right.}$$

$$\simeq \sqrt{\frac{2H}{g}}\,\omega$$

(higher order terms are ignored.)

Therefore, sideways displacement of the particle is

$$\Delta = R\phi - R\omega T = \frac{R}{3}\frac{\varepsilon}{1-\varepsilon}\phi^3 \quad \text{(using (1))}$$

And, $$\Delta \simeq \frac{R}{3}\frac{g}{R\omega^2}\left(\frac{2H}{g}\right)^{3/2}\omega^3 \quad \left[\because \frac{g}{R\omega}\right.$$

$$= \frac{2H}{3}\sqrt{\frac{2H}{g}}\,\omega$$

This result is in conformity with that obtained from calculations based on the Coriolis force.

Note that if the sector OPQ is taken to be triangular in form (which it approximately is) the expression for Δ turns out to be $H\sqrt{(2H/g)}\,\omega$. Clearly, when estimating the difference between two small and nearly equal quantities, *viz.* $R\varphi$ and $R\omega T$, extreme care is necessary.

If the incident takes place at any latitude λ, then in the calculation for sideways deflection, one should replace ω by $\omega\cos\lambda$. The standard result can be recovered proceeding in the same manner as above. Since the earth is moving from west to east, obviously the deflection will be towards east. However, since the orbital plane of the particle must contain the centre of the earth, the point of contact on the surface of the earth will be at a latitude slightly less than λ (meaning an additional southward movement in the northern hemisphere and northward movement in the southern hemisphere) for $0^o < \lambda < 90^o$.

References:

1. R. G. Takwale. P. S. Puranik, *Introduction to Classical Mechanics*, 2nd reprint (1982), p257, Tata McGraw-Hill Publishing Company Ltd., New Delhi.
2. W. G. Rees, *Eur. J. Phys.*, **7 ,** 274-277 (1986)