Euler : Analysis Incarnate*

J.K. Bhattacharjee
S.N.Bose National Centre For Basic Sciences,
Block-JD, Sector-III, Salt Lake,
Kolkata-700098, India

Abstract

On the tomb of Christopher Wren, the architect of St. Paul's cathedral, is the famous Latin inscription whose English rendering is "Visitor, if you want to see his monument, look around". Léonard Euler was a mathematician for whom a similar statement can be made – the cathedral in this case being replaced by the whole wide world. This article discusses a couple of Euler's contribution which all of us have seen in some form or other.

1

This lecture is a tribute to the mathematician Léonard Euler whose third birth centenary is being celebrated this year. We will focus on two of Euler's virtually infinite contributions: his work on certain infinite series and his calculation of the critical load for the buckling of a vertical beam. We begin with the infinite series

$$G(2)=\sum_{n=1}^{\infty}\frac{1}{n^2} \tag{1.1}$$

It was Jacob Bernoulli who tried to obtain a value for $G(2)$ but failed to do. That made the evaluation of $G(2)$ a mathematical challenge and Euler settled the issue in 1734 after beginning with a numerical evaluation in 1730. Euler's technique was to consider the equation

$$\frac{\sin\sqrt{x}}{\sqrt{x}}=0 \tag{1.2}$$

The roots of this equation are $\sqrt{x}=\pm n\pi,\ n=1,2,3,....,$ i.e. $x=n^2\pi^2$. We can now expand $\sin\theta$ in powers of θ and write Eq. (1.2) as

$$\frac{1}{\sqrt{x}}\left[\sqrt{x}-\frac{x^{3/2}}{3!}+\frac{x^{5/2}}{5!}-\frac{x^{7/2}}{7!}+....\right.$$ which is

$$1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+......=0 \tag{1.3}$$

Now if $\alpha_1,\alpha_2,.......\alpha_n$ be the n roots of

$$f(x)=0, \tag{1.4}$$

where $f(x)$ is a polynomial of degree 'n', i.e. $f(x) = a_nx^n + a_{n-1}x^{n-1} + ...a_0$. The roots $\alpha_1,\alpha_2,.....$ satisfy

$$\alpha_1\ \alpha_2.......\alpha_n=(-)^n a_0$$

$$\alpha_1\alpha_2.....\alpha_{n-1}+\alpha_1\alpha_2.....\alpha_{n-2}\alpha_n+.$$

Dividing one by the other

$$\frac{1}{\alpha_n}+\frac{1}{\alpha_{n-1}}+.......+\frac{1}{\alpha_1}=-\frac{a_0}{a_1} \tag{1.5}$$

i.e, we have a formula for the sum of the reciprocals of the roots of $f(x) = 0$. Euler proposed that this well known result for the finite polynomial be extended to the infinite polynomial of Eq (1.3). Using Eq (1.5) for the special case

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of Eq. (1.3), $\sum \frac{1}{roots} = \frac{1}{6}$. Since the roots are $n^2\pi^2$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad (1.6)$$

We now do this in a slightly different form which allows for the evaluation of a number of different sums. We write $p(x) = \frac{\sin x}{x} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + .$ If $f(x)$, the polynomial discussed above of degree 'n' with roots $\alpha_1, \alpha_2, ... \alpha_n$, has the property $f(0) = 1$, then we can write $f(x) = \left(1 - \frac{x}{\alpha_1}\right)\left(1 - \frac{x}{\alpha_2}\right).....\left(1 - \right.$. Euler now boldly asserts that the factorization holds for $p(x)$ (infinite polynomial) as well and since the roots are $\pm n\pi$ (n = 1, 2, 3, ...:)

$$1 - \frac{x}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ...$$

$$. = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \right.$$

$$= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right).. \qquad (1.7)$$

Multiplying out the right hand side, we notice that the coefficient of x^2 is $-\sum \frac{1}{n^2\pi^2}$ and comparing with the left hand side, we get back Eq.(1.6). If we now go one step further and find the coefficient C_4 of x^4 on the right hand side of Eq.(1.7), then we can arrive at $G(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}$.

We note that

$$\pi^4 C_4 = 1 \cdot \left[\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...\right] +$$

$$+ \frac{1}{3^2}\left[\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + ...\right]$$

$$= 1 \cdot \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...\right] + \frac{1}{2^2}$$

$$+ \frac{1}{3^2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + ...\right.$$

$$-1 - \frac{1}{2^2}\left(1 + \frac{1}{2^2}\right) - \frac{1}{3^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right.$$

$$= G(2)\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + ...\right] - \frac{1}{1^4} -$$

$$-1\left[\frac{1}{2^2} + \frac{1}{3^2} + ...\right] - \frac{1}{2^2}\left[\frac{1}{3^2} + \frac{1}{4^2}\right.$$

$$= G(2)^2 - G(4) - \pi^4 C_4 \qquad (1.8)$$

This leads to $G(4) = G(2)^2 - 2\pi^4 C_4$

From the left hand side of Eq. (1.7), $C_4 = \frac{1}{120}$, and thus

$$G(4) = \frac{\pi^4}{90} \qquad (1.9)$$

Comparing the coefficients of x^6 on either side of Eq. (1.7), leads, after some manipulation, to

$$G(6) = \frac{\pi^6}{945} \qquad (1.10)$$

Extending this still further, Euler found

$$G(26) = \sum_{n=1}^{\infty} \frac{1}{n^{26}} = 2^{24} \frac{76977927}{27!} \pi \qquad (1.11)$$

The general expression for $G(2n)$ followed in 1739 . How about $G(2n + 1)$? This is still an unsolved problem. The best that Euler could do was to show that

$$\sum (-)^n \frac{1}{(2n+1)^3} = 1 - \frac{1}{3^3} + \frac{1}{5^3}..... \qquad (1.12)$$

In the course of this investigation, Euler found the connection between the harmonic series and the primes, proving that

$$G(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \prod_p \left(1 - p^{-1}\right)^{-1} \qquad (1.13)$$

This immediately show that the number of primes is infinite. Finally, setting $x = \pi/2$ in Eq. (1.7),

$$\frac{2}{\pi} = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{16}\right)\left(1 - \frac{1}{36}\right).....$$

– a small gem known as the Wallis theorem.

2

In engineering, buckling is a failure mode characterized by the sudden failure of compressive stress. The first quantitative result for the critical buckling load was found by Euler in 1757. This can be considered a forerunner of all subsequent developments in the study of stability. The problem that Euler investigated was the stability of a long slender column under the application of an axial load, i.e, a load whose weight passes through the centre of gravity of the beam. To make progress, Euler had to give a picture of what would be meant by instability. His proposition was that under the loading if there existed a consistent solution for the shape of the beam, a shape which is different from the original but close to shape, then the beam would be called unstable. This was the forerunner of the linear stability analysis that dominates the entire field of nonlinear dynamics. Euler's analysis proceeds as follows. If $v(x)$ is the displacement of the column at the position x (see figure 2), then, for stability, the bending moment has to be equal to the moment of the external load W — the moment being taken about the neutral axis of the beam. The moment of W is $Wv(x)$, while the bending moment according to the Euler-Bernoulli theory is $EI\frac{d^2v}{dx^2}$ where E is Young's modulus and I is the areal moment of inertia about the neutral axis. Then the torque balance gives

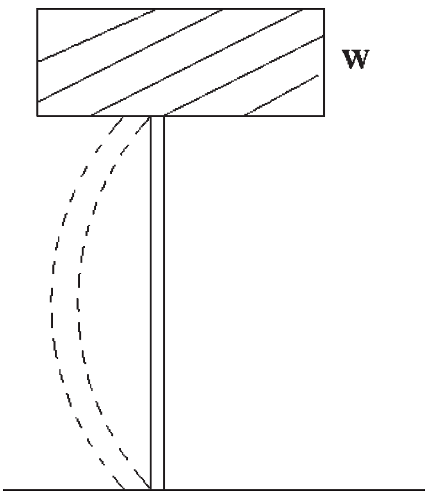

Fig. 1

$$EI\frac{d^2v}{dx^2} + Wv = 0 \tag{2.1}$$

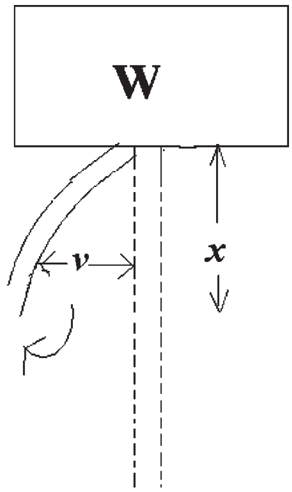

Figure 2

The solution for $v(x)$ can be written as

$$v(x) = A\cos\sqrt{\frac{W}{EI}}\,x + B\sin\sqrt{\frac{W}{EI}}\,x \tag{2.2}$$

The boundary conditions are that the beam is clamped at the two extremes and hence the compatible solutions are of the form

$$v(x) = A\sin\sqrt{\frac{W}{EI}}\,x \tag{2.3}$$

Since $v(L) = 0$, then further constrains the solutions $v(x)$ to be such that

$$\sqrt{\frac{W}{EI}}L = n\pi \tag{2.4}$$

where n is an integer. This means that for a given beam (E,I, and L fixed) W has to be such that

$$W = n^2\pi^2\frac{EI}{L^2} \tag{2.5}$$

A shape close to the original shape exists for n =1 (no intermediate zeroes of $v(x)$) and this gives the critical buckling load W_c as

$$W_c = \frac{\pi^2 EI}{L^2} \tag{2.6}$$

Could a beam buckle under its own weight? Euler turned to this problem in 1757 and the first conclusion was negative. He realized that the answer was unphysical and in a few months found the correct answer. The free body diagram, now takes the form of figure 3.

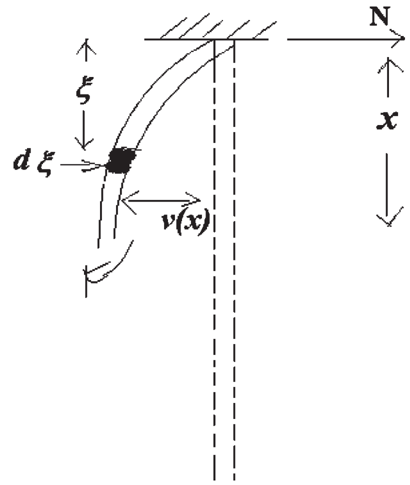

Figure 3

The weight of the beam is q per unit length. The moment of this weight about the section at distance x is found as the integral $q\int_0^x [v(x)-v(\xi)]\,d\xi$. The horizontal reaction at the point of support is N and hence the torque balance condition is

$$EI\frac{d^2v}{dx^2} = -Nx - q\int_0^x [v(x)-v(\xi)]$$

$$= -Nx - qv(x).x + q\int_0^x v(\xi)\,d\xi \tag{2.7}$$

$$= -Nx - q\int_0^x \xi v'(\xi)\,d\xi$$

Differentiating twice with respect to x,

$$EI\frac{d^4v}{dx^4} + q\left[x\frac{d^2v}{dx^2} + \frac{dv}{dx}\right] = 0 \tag{2.8}$$

The pinned-pinned boundary conditions are $v(x) = \frac{d^2}{dx^2} = 0$ at the two ends x = 0 and x = L. The lowest value of q for which a solution exists satisfying these boundary conditions is

$$q = 18.6\frac{EI}{L^3} \tag{2.9}$$

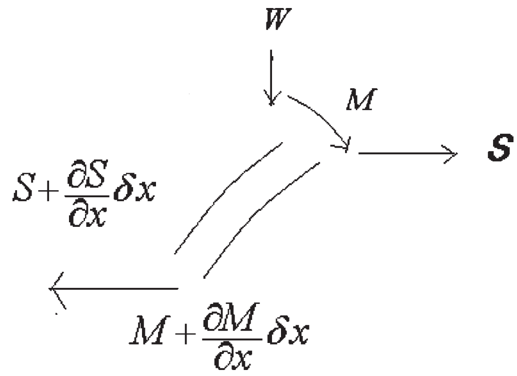

Figure 4

Thus Euler set the stage for what would become an almost all-encompassing field of instabilities and pattern formation. In today's parlance, the buckling problem under an external load would be formulated as a problem in dynamics. We would focus on a small part δx of the beam and $v(x)$ would be deviation from the vertical equilibrium. At x, the displacement of the beam is $v(x)$, the bending moment M and the shear S , while at $x + \delta x$, the displacement is $v + \frac{\partial v}{\partial x}dx$, the bending moment $M + \frac{\partial M}{\partial x}dx$ and the shear $S + \frac{\partial S}{\partial x}dx$. Torque balance on the section yields

$$\frac{\partial M}{\partial x} + W\frac{\partial v}{\partial x} + S = 0 \tag{2.10}$$

Newton's law leads to

$$\rho\frac{\partial^2 v}{\partial t^2} = \frac{\partial S}{\partial x} = \frac{\partial^2 M}{\partial x^2} - W\frac{\partial^2 v}{\partial x^2} \tag{2.11}$$

on using Eq.(2.10). Since the bending moment $M = EI\frac{\partial^2 v}{\partial x^2}$, we have

$$\rho\frac{\partial^2 v}{\partial t^2} = -EI\,\frac{\partial^4 v}{\partial x^4} - W\,\frac{\partial^2 v}{\partial x^2} \qquad (2.12)$$

One can now find the characteristic frequencies and ask the question for what value of *W* does the lowest frequency become imaginary – this would signal an unbounded increase in the deviation *v* from equilibrium – that would imply the original equilibrium (vertical position) has become unstable. As expected the critical load works out the same as that shown in Eq.(2.6). Interestingly enough, the buckling also gives the first indication of spontaneous symmetry breaking — another phenomenon of wide interest.